Example 21.1 A Tank of Helium

A tank used for filling helium balloons has a volume of 0.300 m^3 and contains 2.00 mol of helium gas at 20.0°C. Assume the helium behaves like an ideal gas.

(A) What is the total translational kinetic energy of the gas molecules?

SOLUTION

Conceptualize Imagine a microscopic model of a gas in which you can watch the molecules move about the container more rapidly as the temperature increases.

Categorize We evaluate parameters with equations developed in the preceding discussion, so this example is a substitution problem.

Use Equation 21.6 with $n = 2.00$ mol and $T = 293$ K:

$$K_{\text{tot trans}} = \tfrac{3}{2}nRT = \tfrac{3}{2}(2.00\text{ mol})(8.31\text{ J/mol}\cdot\text{K})(293\text{ K})$$
$$= \boxed{7.30 \times 10^3\text{ J}}$$

(B) What is the average kinetic energy per molecule?

SOLUTION

Use Equation 21.4:

$$\tfrac{1}{2}m_0v^2 = \tfrac{3}{2}k_BT = \tfrac{3}{2}(1.38 \times 10^{-23}\text{ J/K})(293\text{ K})$$
$$= \boxed{6.07 \times 10^{-21}\text{ J}}$$

WHAT IF? What if the temperature is raised from 20.0°C to 40.0°C? Because 40.0 is twice as large as 20.0, is the total translational energy of the molecules of the gas twice as large at the higher temperature?

Answer The expression for the total translational energy depends on the temperature, and the value for the temperature must be expressed in kelvins, not in degrees Celsius. Therefore, the ratio of 40.0 to 20.0 is *not* the appropriate ratio. Converting the Celsius temperatures to kelvins, 20.0°C is 293 K and 40.0°C is 313 K. Therefore, the total translational energy increases by a factor of only 313 K/293 K = 1.07.

Example 21.2 Heating a Cylinder of Helium

A cylinder contains 3.00 mol of helium gas at a temperature of 300 K.

(A) If the gas is heated at constant volume, how much energy must be transferred by heat to the gas for its temperature to increase to 500 K?

SOLUTION

Conceptualize Run the process in your mind with the help of the piston–cylinder arrangement in Active Figure 19.12. Imagine that the piston is clamped in position to maintain the constant volume of the gas.

Categorize We evaluate parameters with equations developed in the preceding discussion, so this example is a substitution problem.

Use Equation 21.8 to find the energy transfer: $Q_1 = nC_V \Delta T$

Substitute the given values: $Q_1 = (3.00 \text{ mol})(12.5 \text{ J/mol} \cdot \text{K})(500 \text{ K} - 300 \text{ K})$

$= \boxed{7.50 \times 10^3 \text{ J}}$

(B) How much energy must be transferred by heat to the gas at constant pressure to raise the temperature to 500 K?

SOLUTION

Use Equation 21.9 to find the energy transfer: $Q_2 = nC_P \Delta T$

Substitute the given values: $Q_2 = (3.00 \text{ mol})(20.8 \text{ J/mol} \cdot \text{K})(500 \text{ K} - 300 \text{ K})$

$= \boxed{12.5 \times 10^3 \text{ J}}$

This value is larger than Q_1 because of the transfer of energy out of the gas by work to raise the piston in the constant pressure process.

Example 21.3 A Diesel Engine Cylinder

Air at 20.0°C in the cylinder of a diesel engine is compressed from an initial pressure of 1.00 atm and volume of 800.0 cm^3 to a volume of 60.0 cm^3. Assume air behaves as an ideal gas with $\gamma = 1.40$ and the compression is adiabatic. Find the final pressure and temperature of the air.

SOLUTION

Conceptualize Imagine what happens if a gas is compressed into a smaller volume. Our discussion above and Figure 21.5 tell us that the pressure and temperature both increase.

Categorize We categorize this example as a problem involving an adiabatic process.

[2]In the adiabatic free expansion discussed in Section 20.6, the temperature remains constant. In this unique process, no work is done because the gas expands into a vacuum. In general, the temperature decreases in an adiabatic expansion in which work is done.

21.3 *cont.*

Analyze Use Equation 21.19 to find the final pressure:

$$P_f = P_i\left(\frac{V_i}{V_f}\right)^{\gamma} = (1.00\text{ atm})\left(\frac{800.0\text{ cm}^3}{60.0\text{ cm}^3}\right)^{1.40}$$

$$= \boxed{37.6\text{ atm}}$$

Use the ideal gas law to find the final temperature:

$$\frac{P_iV_i}{T_i} = \frac{P_fV_f}{T_f}$$

$$T_f = \frac{P_fV_f}{P_iV_i}T_i = \frac{(37.6\text{ atm})(60.0\text{ cm}^3)}{(1.00\text{ atm})(800.0\text{ cm}^3)}(293\text{ K})$$

$$= 826\text{ K} = \boxed{553^\circ\text{C}}$$

Finalize The temperature of the gas increases by a factor of 826 K/293 K = 2.82. The high compression in a diesel engine raises the temperature of the gas enough to cause the combustion of fuel without the use of spark plugs.

21.4 *cont.*

Evaluate $k_B T$ in the exponent:

$$k_B T = (1.38 \times 10^{-23}\ \text{J/K})(2\,500\ \text{K})\left(\frac{1\ \text{eV}}{1.60 \times 10^{-19}\ \text{J}}\right) = 0.216\ \text{eV}$$

Substitute this value into Equation (1):

$$\frac{n_V(E_2)}{n_V(E_1)} = e^{-1.50\ \text{eV}/0.216\ \text{eV}} = e^{-6.96} = \boxed{9.52 \times 10^{-4}}$$

Finalize This result indicates that at $T = 2\,500$ K, only a small fraction of the atoms are in the higher energy level. In fact, for every atom in the higher energy level, there are about 1 000 atoms in the lower level. The number of atoms in the higher level increases at even higher temperatures, but the distribution law specifies that at equilibrium there are always more atoms in the lower level than in the higher level.

WHAT IF? What if the energy levels in Figure 21.9 were closer together in energy? Would that increase or decrease the fraction of the atoms in the upper energy level?

Answer If the excited level is lower in energy than that in Figure 21.9, it would be easier for thermal agitation to excite atoms to this level and the fraction of atoms in this energy level would be larger, which we can see mathematically by expressing Equation (1) as

$$r_2 = e^{-(E_2 - E_1)/k_B T}$$

where r_2 is the ratio of atoms having energy E_2 to those with energy E_1. Differentiating with respect to E_2, we find

$$\frac{dr_2}{dE_2} = \frac{d}{dE_2}\left[e^{-(E_2 - E_1)/k_B T}\right] = -\frac{1}{k_B T}e^{-(E_2 - E_1)/k_B T} < 0$$

Because the derivative has a negative value, as E_2 decreases, r_2 increases.

Example 21.4 Thermal Excitation of Atomic Energy Levels

As discussed in Section 21.4, atoms can occupy only certain discrete energy levels. Consider a gas at a temperature of 2 500 K whose atoms can occupy only two energy levels separated by 1.50 eV, where 1 eV (electron volt) is an energy unit equal to 1.60×10^{-19} J (Fig. 21.9). Determine the ratio of the number of atoms in the higher energy level to the number in the lower energy level.

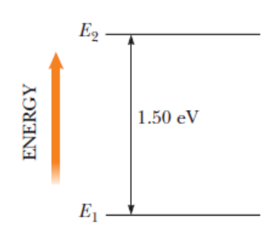

Figure 21.9 (Example 21.4) Energy-level diagram for a gas whose atoms can occupy two energy states.

SOLUTION

Conceptualize In your mental representation of this example, remember that only two possible states are allowed for the system of the atom. Figure 21.9 helps you visualize the two states on an energy-level diagram. In this case, the atom has two possible energies, E_1 and E_2, where $E_1 < E_2$.

Categorize We categorize this example as one in which we apply the Boltzmann distribution law to a quantized system.

Analyze Set up the ratio of the number of atoms in the higher energy level to the number in the lower energy level and use Equation 21.23 to express each number:

$$(1) \quad \frac{n_V(E_2)}{n_V(E_1)} = \frac{n_0 e^{-E_2/k_B T}}{n_0 e^{-E_1/k_B T}} = e^{-(E_2 - E_1)/k_B T}$$

Example 21.5 A System of Nine Particles

Nine particles have speeds of 5.00, 8.00, 12.0, 12.0, 12.0, 14.0, 14.0, 17.0, and 20.0 m/s.

(A) Find the particles' average speed.

SOLUTION

Conceptualize Imagine a small number of particles moving in random directions with the few speeds listed.

Categorize Because we are dealing with a small number of particles, we can calculate the average speed directly.

Analyze Find the average speed of the particles by dividing the sum of the speeds by the total number of particles:

$$v_{avg} = \frac{(5.00 + 8.00 + 12.0 + 12.0 + 12.0 + 14.0 + 14.0 + 17.0 + 20.0)\text{ m/s}}{9}$$

$$= \boxed{12.7 \text{ m/s}}$$

(B) What is the rms speed of the particles?

SOLUTION

Find the average speed squared of the particles by dividing the sum of the speeds squared by the total number of particles:

$$\overline{v^2} = \frac{(5.00^2 + 8.00^2 + 12.0^2 + 12.0^2 + 12.0^2 + 14.0^2 + 14.0^2 + 17.0^2 + 20.0^2)\text{ m}^2/\text{s}^2}{9}$$

$$= 178 \text{ m}^2/\text{s}^2$$

Find the rms speed of the particles by taking the square root:

$$v_{rms} = \sqrt{\overline{v^2}} = \sqrt{178 \text{ m}^2/\text{s}^2} = \boxed{13.3 \text{ m/s}}$$

(C) What is the most probable speed of the particles?

SOLUTION

Three of the particles have a speed of 12.0 m/s, two have a speed of 14.0 m/s, and the remaining four have different speeds. Hence, the most probable speed v_{mp} is 12.0 m/s.

Finalize Compare this example, in which the number of particles is small and we know the individual particle speeds, with the next example.

Example 21.6 Molecular Speeds in a Hydrogen Gas

A 0.500-mol sample of hydrogen gas is at 300 K.

(A) Find the average speed, the rms speed, and the most probable speed of the hydrogen molecules.

SOLUTION

Conceptualize Imagine a huge number of particles in a real gas, all moving in random directions with different speeds.

Categorize We cannot calculate the averages as was done in Example 21.5 because the individual speeds of the particles are not known. We are dealing with a very large number of particles, however, so we can use the Maxwell-Boltzmann speed distribution function.

Analyze Use Equation 21.26 to find the average speed:

$$v_{avg} = 1.60\sqrt{\frac{k_BT}{m_0}} = 1.60\sqrt{\frac{(1.38 \times 10^{-23}\text{ J/K})(300\text{ K})}{2(1.67 \times 10^{-27}\text{ kg})}}$$

$$= \boxed{1.78 \times 10^3\text{ m/s}}$$

Use Equation 21.25 to find the rms speed:

$$v_{rms} = 1.73\sqrt{\frac{k_BT}{m_0}} = 1.73\sqrt{\frac{(1.38 \times 10^{-23}\text{ J/K})(300\text{ K})}{2(1.67 \times 10^{-27}\text{ kg})}}$$

$$= \boxed{1.93 \times 10^3\text{ m/s}}$$

Use Equation 21.27 to find the most probable speed:

$$v_{mp} = 1.41\sqrt{\frac{k_BT}{m_0}} = 1.41\sqrt{\frac{(1.38 \times 10^{-23}\text{ J/K})(300\text{ K})}{2(1.67 \times 10^{-27}\text{ kg})}}$$

$$= \boxed{1.57 \times 10^3\text{ m/s}}$$

(B) Find the number of molecules with speeds between 400 m/s and 401 m/s.

SOLUTION

Use Equation 21.24 to evaluate the number of molecules in a narrow speed range between v and $v + dv$:

$$(1)\quad N_v\,dv = 4\pi N\left(\frac{m_0}{2\pi k_BT}\right)^{3/2} v^2 e^{-m_0v^2/2k_BT}\,dv$$

Evaluate the constant in front of v^2:

$$4\pi N\left(\frac{m_0}{2\pi k_BT}\right)^{3/2} = 4\pi n N_A\left(\frac{m_0}{2\pi k_BT}\right)^{3/2}$$

$$= 4\pi(0.500\text{ mol})(6.02 \times 10^{23}\text{ mol}^{-1})\left[\frac{2(1.67 \times 10^{-27}\text{ kg})}{2\pi(1.38 \times 10^{-23}\text{ J/K})(300\text{ K})}\right]^{3/2}$$

$$= 1.74 \times 10^{14}\text{ s}^3/\text{m}^3$$

Evaluate the exponent of e that appears in Equation (1):

$$-\frac{m_0v^2}{2k_BT} = -\frac{2(1.67 \times 10^{-27}\text{ kg})(400\text{ m/s})^2}{2(1.38 \times 10^{-23}\text{ J/K})(300\text{ K})} = -0.064\,5$$

Evaluate $N_v\,dv$ using Equation (1):

$$N_v\,dv = (1.74 \times 10^{14}\text{ s}^3/\text{m}^3)(400\text{ m/s})^2 e^{-0.064\,5}(1\text{ m/s})$$

$$= \boxed{2.61 \times 10^{19}\text{ molecules}}$$

Finalize In this evaluation, we could calculate the result without integration because dv = 1 m/s is much smaller than v = 400 m/s. Had we sought the number of particles between, say, 400 m/s and 500 m/s, we would need to integrate Equation (1) between these speed limits.